Suggested solutions of assignment 3 :

(1) To see $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, we use Prop. 1 in hand-out 3.

We need to check $\nabla f(x,y,z)\neq 0$ for all $(x,y,z)\in\mathbb{R}^3$ s.t. $f(x,y,z)=1$

where $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}$

$$\nabla f(x,y,z)=2\left(\frac{x}{a^2},\frac{y}{b^2},\frac{-z}{c^2}\right)$$

$$\nabla f(x,y,z)=0 \text{ iff } (x,y,z)=0$$

And $f(x,y,z)=1 \Rightarrow (x,y,z)\neq 0$

So 1 is a regular value of f. Then by Prop. 1,

$M=\{\vec{x}\in\mathbb{R}^3 \mid f(\vec{x})=1\}$ is a regular surface.

To see $X(u,v)=(a\cosh u\cos v, b\cosh u\sin v, c\sinh u)$ is a parametrization. We should first see that

$$\frac{(a\cosh u\cos v)^2}{a^2}+\frac{(b\cosh u\sin v)^2}{b^2}-\frac{(c\sinh u)^2}{c^2}=1.$$

And $X_u\times X_v\neq 0$

$$\left[\begin{aligned} X_u&=(a\sinh u\cos v, b\sinh u\sin v, c\cosh u)\\ X_v&=(-a\cosh u\sin v, b\cosh u\cos v, 0)\\ X_u\times X_v&=(-bc\cosh^2 u\cos v, -ac\cosh^2 u\sin v, ab\sinh u\cdot\cosh u)\\ &\neq 0\end{aligned}\right]$$

One can check the domain $\mathbb{R} \times (0, 2\pi)$ on (u,v)-plane s.t X is 1-1.

(2) $\alpha(s) = (\cos s, \sin s)$

$w(s) = (-\sin s, \cos s, 1)$

$$X(s,v) = (\cos s, \sin s, 0) + v(-\sin s, \cos s, 1)$$
$$= (\cos s - v\sin s, \sin s + v\cos s, v)$$

Need to check
$$(\cos s - v\sin s)^2 + (\sin s + v\cos s)^2 - v^2 = 1.$$

Onto : We may write $X(s,v) = \left(\sqrt{1+v^2}\cos(s+\theta), \sqrt{1+v^2}\sin(s+\theta), v\right)$

where $\theta = \arccos\frac{1}{\sqrt{1+v^2}}$. $\left(\in [0, \frac{\pi}{2}]\right)$

Given any (x_0, y_0, z_0) s.t $x_0^2 + y_0^2 - z_0^2 = 1$.

Let $v_0 = z_0$

$$s_0 = \arccos\frac{x_0}{\sqrt{1+z_0^2}} - \theta_0 \quad \left(\theta_0 = \arccos\frac{1}{\sqrt{1+z_0^2}}\right)$$

then $X(s_0, v_0) = (x_0, y_0, z_0)$.

Not 1-1 : Since $s = 0$ or 2π, $X(0, v) = X(2\pi, v)$.

Full rank (rank 2) :
for $0 < s < 2\pi$, $v \in \mathbb{R}$

$$X_s = (-\sin s - v\cos s, \cos s - v\sin s, 0)$$
$$X_v = (-\sin s, \cos s, 1)$$
$$X_s \times X_v = (\cos s - v\sin s, \sin s + v\cos s, -v) \neq 0.$$

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(3).

$$X(u,v) = \left(u,\ \cosh u \cos v,\ \cosh u \sin v \right)$$

$$\forall\ (u,v) \in \mathbb{R} \times [0, 2\pi).$$

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(4) To see $X(u,v)$ is regular for $u^2+v^2<3$, we need to show X is 1-1 and $X_u \times X_v \neq 0$.

One can compute $X_u \times X_v = \left(-2u(u^2+v^2+1),\ 2v(u^2+v^2+1),\ 1-(u^2+v^2)\right)$

$$\neq 0.$$

1-1: Assume we have

$$\begin{cases} u_1 - \frac{u_1^3}{3} + u_1 v_1^2 = u_2 - \frac{u_2^3}{3} + u_2 v_2^2 & (1) \\ v_1 - \frac{v_1^3}{3} + v_1 u_1^2 = v_2 - \frac{v_2^3}{3} + v_2 u_2^2 & (2) \\ u_1^2 - v_1^2 = u_2^2 - v_2^2 & (3) \end{cases}$$

By (1),

$$u_1 - u_2 - \frac{1}{3}(u_1^3 - u_2^3) + u_1 v_1^2 - u_2 v_2^2 = 0. \qquad (4)$$

By (3), $v_1^2 = u_1^2 - u_2^2 + v_2^2$ (5)

Plug (5) into (4),

$$u_1 - u_2 - \frac{1}{3}(u_1 - u_2)(u_1^2 + u_1 \cdot u_2 + u_2^2) + u_1(u_1^2 - u_2^2 + v_2^2) - u_2 v_2^2 = 0$$

$$\Rightarrow \quad (u_1-u_2)\left(1-\tfrac{1}{3}u_1^2-\tfrac{1}{3}u_1u_2-\tfrac{1}{3}u_2^2+u_1^2+u_1\cdot u_2+v_2^2\right)=0$$

$$\Rightarrow \quad (u_1-u_2)\left(1+\tfrac{2}{3}u_1^2+\tfrac{2}{3}u_1\cdot u_2-\tfrac{1}{3}u_2^2+v_2^2\right)=0.$$

Note that

$$1+\tfrac{2}{3}u_1^2+\tfrac{2}{3}u_1\cdot u_2-\tfrac{1}{3}u_2^2+v_2^2 > \tfrac{2}{3}u_1^2+\tfrac{2}{3}u_1u_2+\tfrac{4}{3}v_2^2$$

Since $1 > \frac{1}{3}v_2^2+\frac{1}{3}u_2^2$. by ③

$$= \tfrac{1}{3}(u_1+u_2)^2+v_2^2+\tfrac{1}{3}v_1^2 \geq 0.$$

$\Rightarrow \quad u_1 = u_2$

Similarly, $v_1 = v_2$.

$$X(\sqrt{3},0)=X(-\sqrt{3},0)=(0,0,3).$$

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